Math 261 - Homework Set 0

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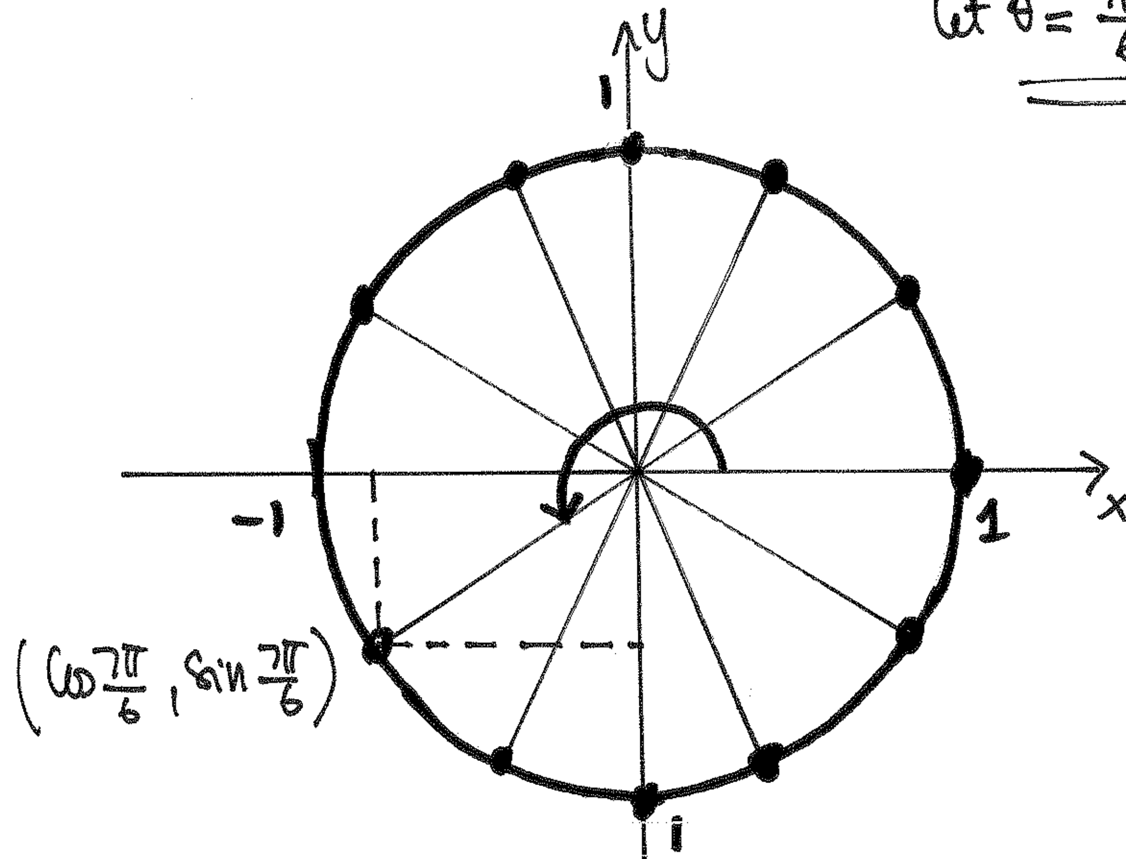
Let $\theta = \frac{7\pi}{6}$.



Using the fact that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{6} = \frac{1}{2}$, and the picture above, we obtain the following

$$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

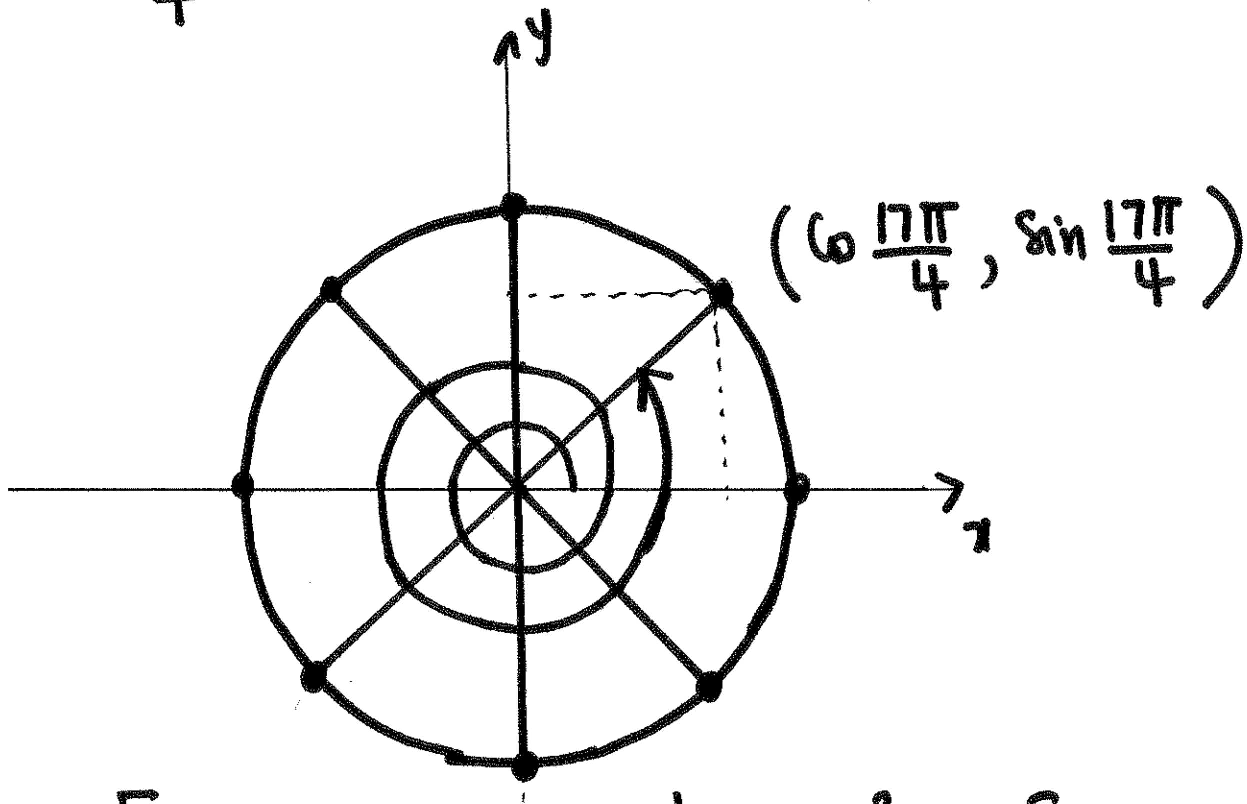
$$\sin\frac{7\pi}{6} = -\frac{1}{2}$$

$$\tan\frac{7\pi}{6} = \frac{\sin\frac{7\pi}{6}}{\cos\frac{7\pi}{6}} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1\,\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot\frac{7\pi}{6} = \frac{1}{\tan\frac{7\pi}{6}} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$

$$\sec\frac{7\pi}{6} = \frac{1}{\cos\frac{7\pi}{6}} = -\frac{2}{\sqrt{3}} = -\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}, \quad \csc\frac{7\pi}{6} = \frac{1}{-1/2} = -2.$$

Let $\theta = \frac{17\pi}{4}$



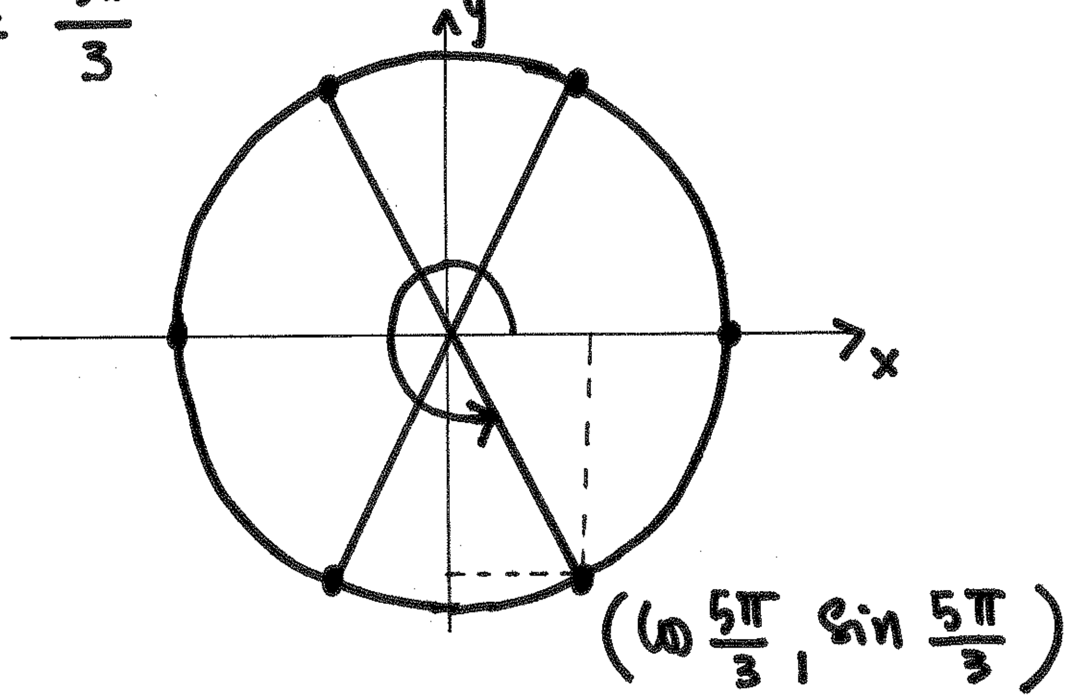
$\cos\frac{17\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $\sec\frac{17\pi}{4} = \frac{1}{\cos\frac{17\pi}{4}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\sin\frac{17\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $\csc\frac{17\pi}{4} = \frac{1}{\sin\frac{17\pi}{4}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\tan\frac{17\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$

$\cot\frac{17\pi}{4} = \frac{1}{\tan\frac{17\pi}{4}} = \frac{1}{1} = 1.$

Let $\theta = \frac{5\pi}{3}$



$$\cos\frac{5\pi}{3} = \frac{1}{2}$$

$$\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\tan\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}\cdot\frac{2}{1} = -\sqrt{3}$$

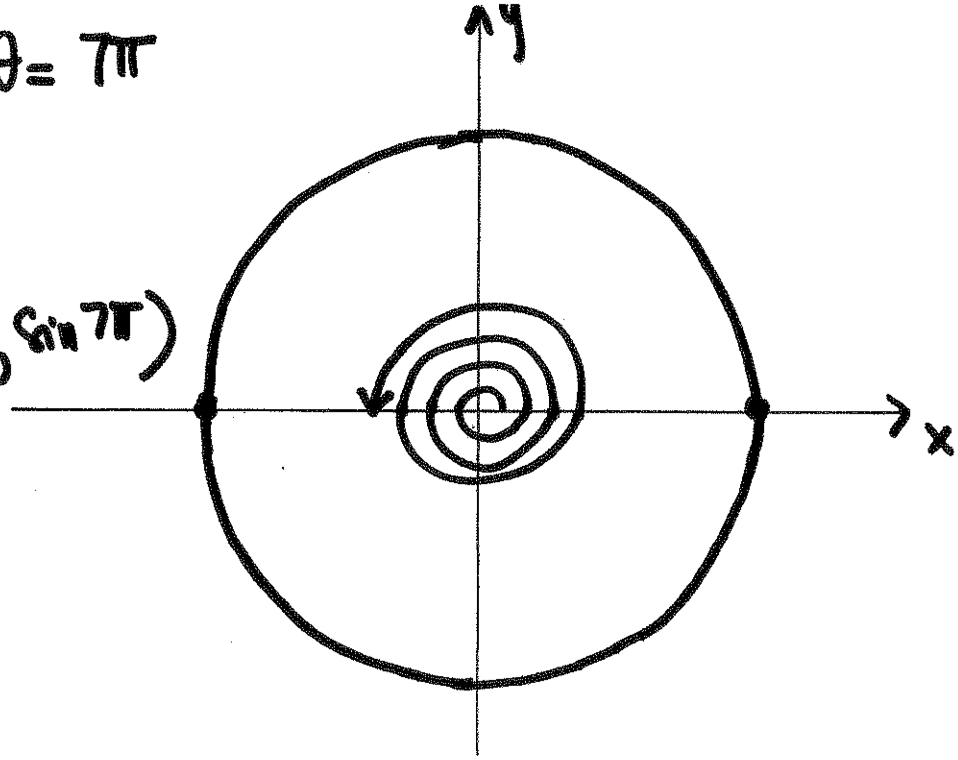
$$\cot\frac{5\pi}{3} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec\frac{5\pi}{3} = 2$$

$$\csc\frac{5\pi}{3} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

Let $\theta = 7\pi$

$(\cos 7\pi, \sin 7\pi)$



$\cos 7\pi = -1$

$\sin 7\pi = 0$

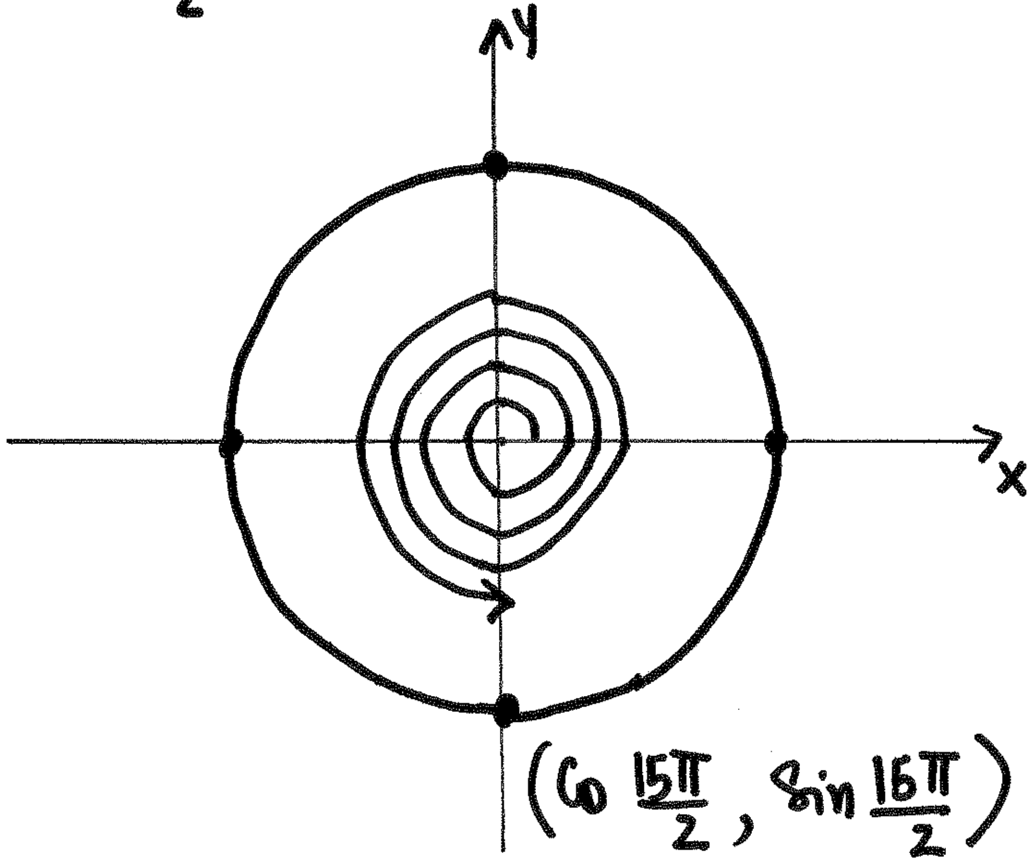
$\tan 7\pi = 0$

$\cot 7\pi$ is undefined

$\sec 7\pi = -1$

$\csc 7\pi$ is undefined.

let $\theta = \frac{15\pi}{2}$



$\cos\frac{15\pi}{2} = 0$

$\sin\frac{16\pi}{2} = -1$

$\tan\frac{15\pi}{2}$ is undefined

$\cot\frac{15\pi}{2} = 0$

$\sec\frac{15\pi}{2}$ is undefined

$\csc\frac{15\pi}{2} = -1$

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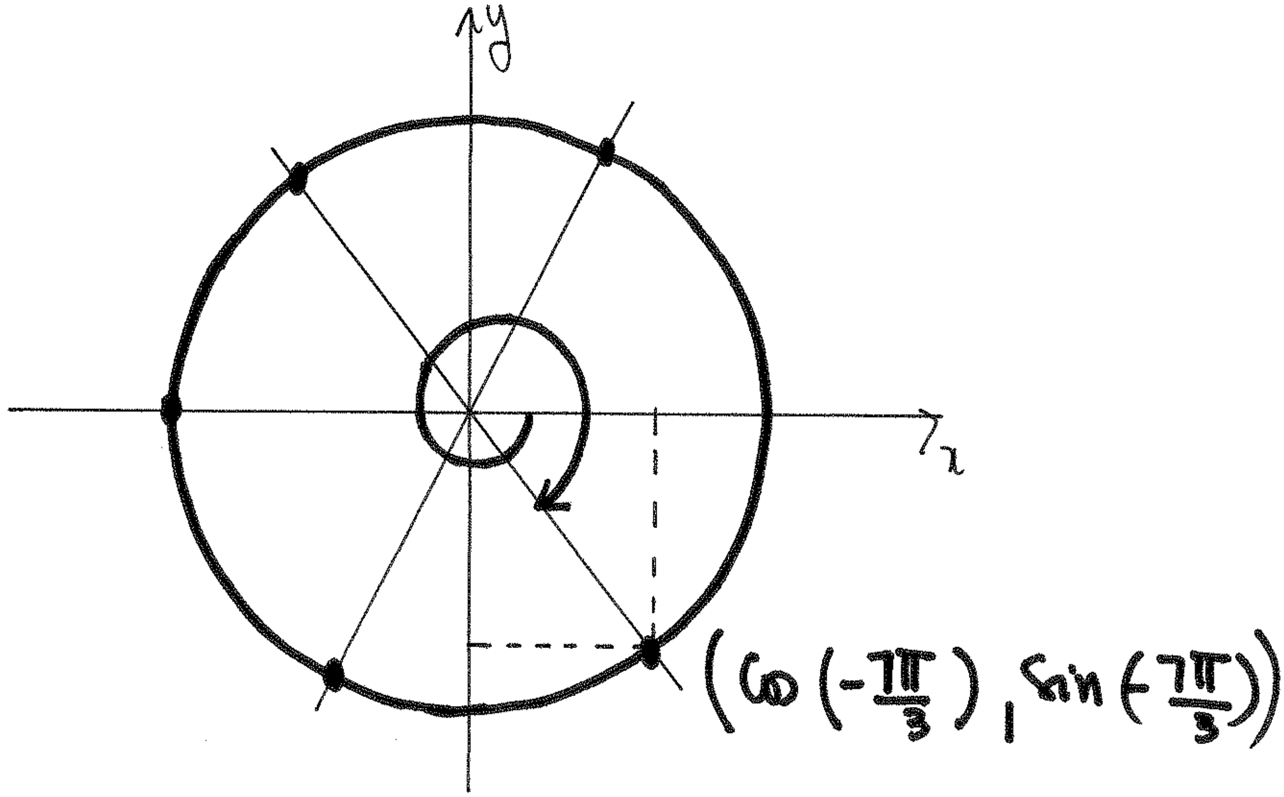


$\cos\left(-\frac{7\pi}{3}\right) = \frac{1}{2}$

$\sin\left(-\frac{7\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

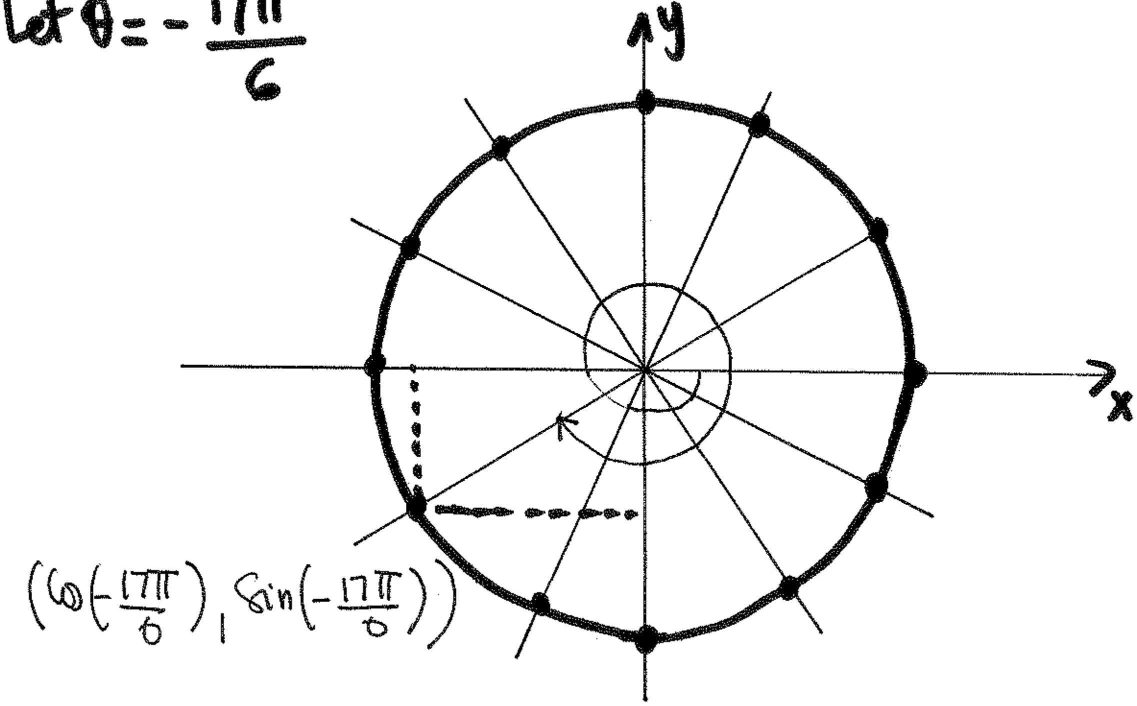
$\tan\left(-\frac{7\pi}{3}\right) = -\frac{\sqrt{3}}{2} \cdot 2 = -\sqrt{3}$

$\cot\left(-\frac{7\pi}{3}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$\sec\left(-\frac{7\pi}{3}\right) = 2$

$\csc\left(-\frac{7\pi}{3}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

Let $\theta = -\frac{17\pi}{6}$



$\cos\left(-\frac{17\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\sin\left(-\frac{17\pi}{6}\right) = -\frac{1}{2}$

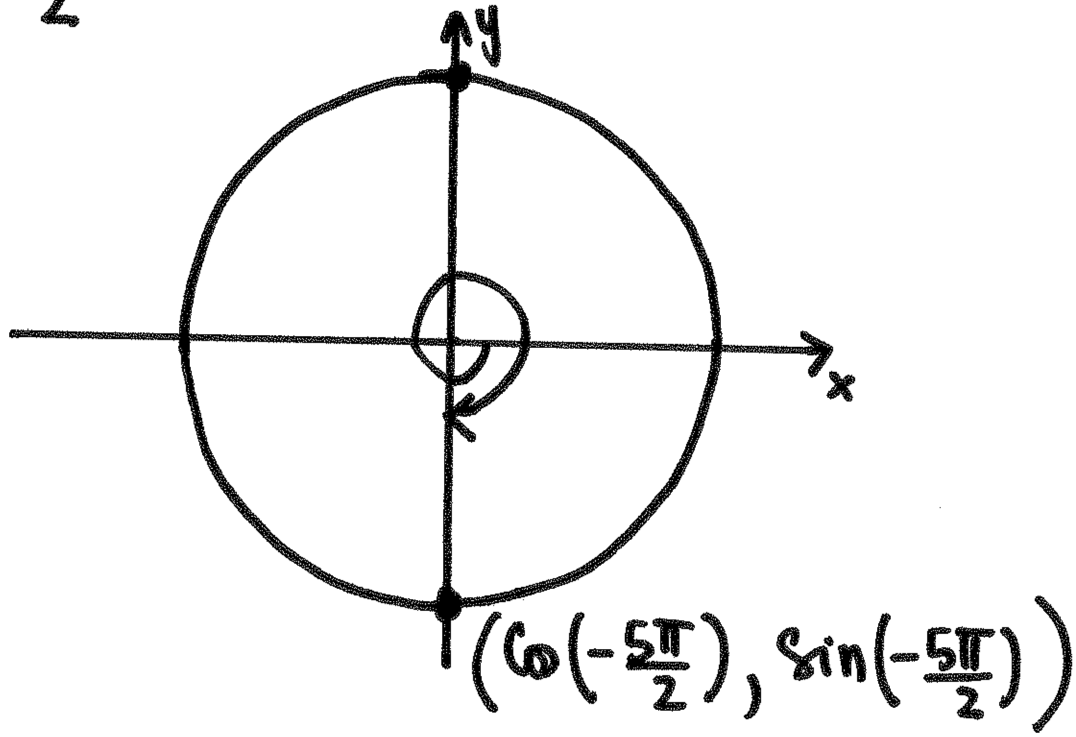
$\tan\left(-\frac{17\pi}{6}\right) = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\cot\left(-\frac{17\pi}{6}\right) = \sqrt{3}$

$\sec\left(-\frac{17\pi}{6}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

$\csc\left(-\frac{17\pi}{6}\right) = -2$.

Let $\theta = -\frac{5\pi}{2}$



$\cos\left(-\frac{5\pi}{2}\right) = 0$

$\sin\left(-\frac{5\pi}{2}\right) = -1$

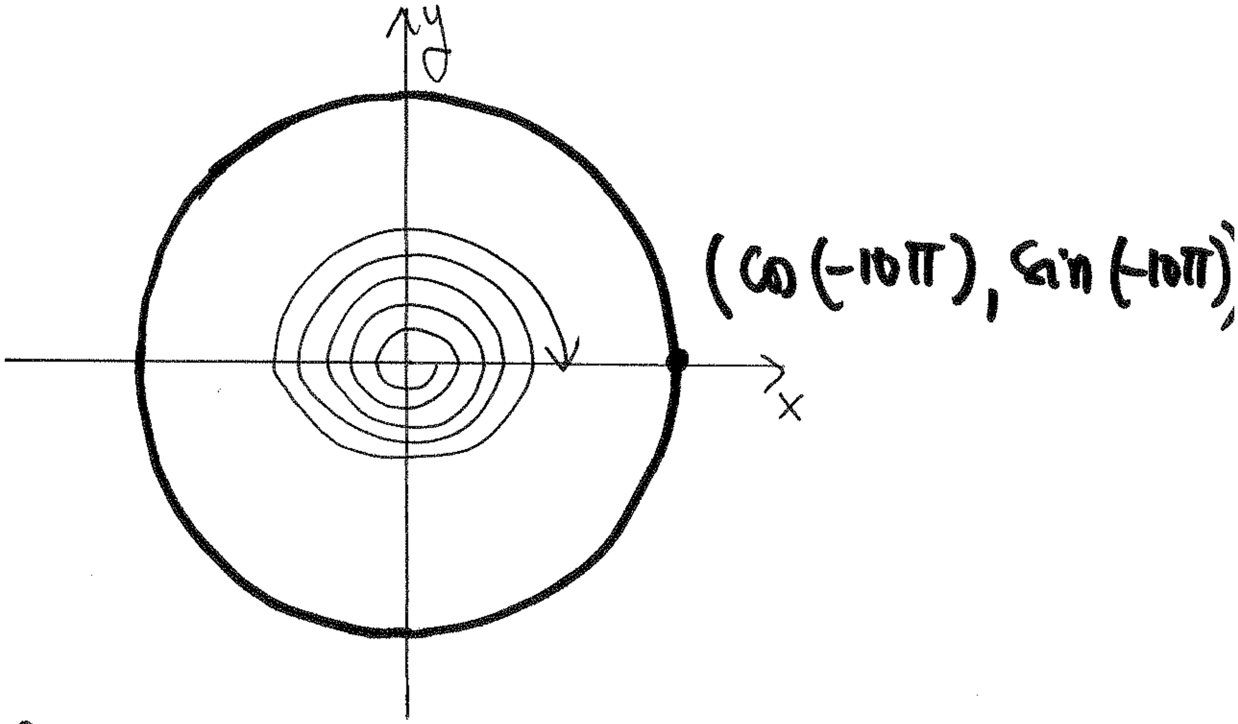
$\tan\left(-\frac{5\pi}{2}\right)$ is undefined

$\cot\left(-\frac{5\pi}{2}\right) = 0$

$\sec\left(-\frac{5\pi}{2}\right)$ is undefined

$\csc\left(-\frac{5\pi}{2}\right) = -1$

$\theta = -10\pi$



$\cos(-10\pi) = 1$

$\sin(-10\pi) = 0$

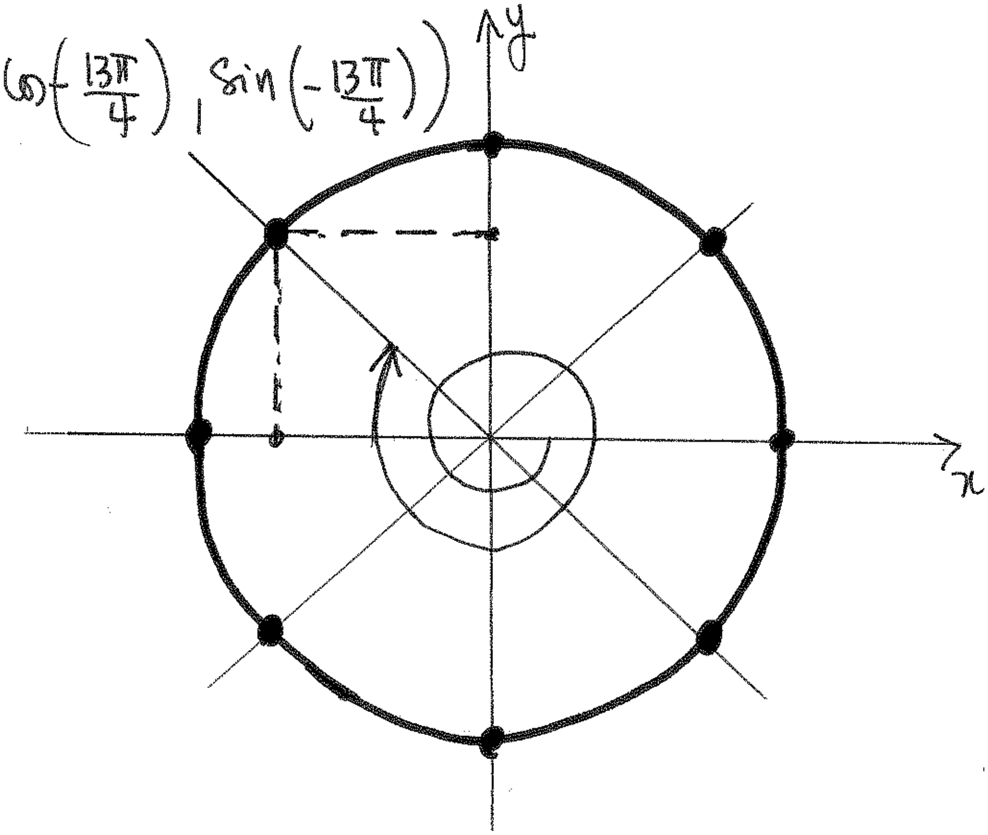
$\tan(-10\pi) = 0$

$\cot(-10\pi)$ is undefined

$\sec(-10\pi) = 1$

$\csc(-10\pi)$ is not defined.

$\left(\cos\left(-\frac{13\pi}{4}\right), \sin\left(-\frac{13\pi}{4}\right)\right)$



$$\cos\left(-\frac{13\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\sin\left(-\frac{13\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{13\pi}{4}\right) = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$$

$$\cot\left(-\frac{13\pi}{4}\right) = \frac{1}{\tan\left(-\frac{13\pi}{4}\right)} = -1$$

$$\sec\left(-\frac{13\pi}{4}\right) = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\csc\left(-\frac{13\pi}{4}\right) = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

3 (a)

$$\frac{\pi}{3}+\frac{\pi}{4}=\frac{4\pi}{12}+\frac{3\pi}{12}=\frac{7\pi}{12}.$$

3 (b)

$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right)&=\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\&=\sin\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos\frac{\pi}{3}\\&=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\&=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\&=\frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right)&=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\&=\cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\&=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\&=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}\\&=\frac{\sqrt{2}-\sqrt{6}}{4}.\end{aligned}$$

$$\tan\left(\frac{7\pi}{12}\right)=\frac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)}=\frac{\sqrt{6}+\sqrt{2}}{4}\cdot\frac{4}{\sqrt{2}-\sqrt{6}}=\frac{\sqrt{6}+\sqrt{2}}{\sqrt{2}-\sqrt{6}}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{(\sqrt{2}-\sqrt{6})} \frac{(\sqrt{2}+\sqrt{6})}{(\sqrt{2}+\sqrt{6})} = \frac{(\sqrt{6}+\sqrt{2})^2}{2-6}$$

$$= \frac{6+2\sqrt{12}+2}{-4} = \frac{8+2\sqrt{12}}{-4} = \frac{8+4\sqrt{3}}{-4}$$

$$= -(2+\sqrt{3}) = -2-\sqrt{3}.$$

th $\tan\left(\frac{7\pi}{12}\right) = -2-\sqrt{3}$.

4 (a) First, we consider $1+\cot^2(x)$.

Since $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$ then

$$1+\cot^2(x) = 1+\left(\frac{\cos x}{\sin x}\right)^2$$

$$= 1+\frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= \frac{1}{\sin^2(x)}$$

We conclude that $\boxed{1+\cot^2(x) = \dfrac{1}{\sin^2 x}}$

4(b) First, we consider $1+\tan^2(x)$.

Since $\tan(x) = \dfrac{\sin x}{\cos x}$ then $1+\tan^2(x) = 1+\left(\dfrac{\sin x}{\cos x}\right)^2$

Next,

$$1+\tan^2(x) = 1+\frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \left(\frac{1}{\cos x}\right)^2$$

$$= \sec^2(x).$$

We conclude that

$$\boxed{1+\tan^2 x = \sec^2(x)}$$

4(c) We consider $\dfrac{\cos x}{1+\sin x} + \tan x$.

Next, $\dfrac{\cos x}{1+\sin x} + \tan x = \dfrac{\cos x}{1+\sin x} + \dfrac{\sin x}{\cos x}$

$$= \frac{(\cos x)(\cos x)}{(1+\sin x)(\cos x)} + \frac{\sin x\,(1+\sin x)}{\cos x\,(1+\sin x)}$$

(common denominator)

$$= \frac{\cos^2 x}{\cos x(1+\sin x)} + \frac{\sin x + \sin^2 x}{\cos x(1+\sin x)}$$

$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos x\,(1+\sin x)}$$

$$= \frac{1+\sin x}{\cos x\,(1+\sin x)}$$

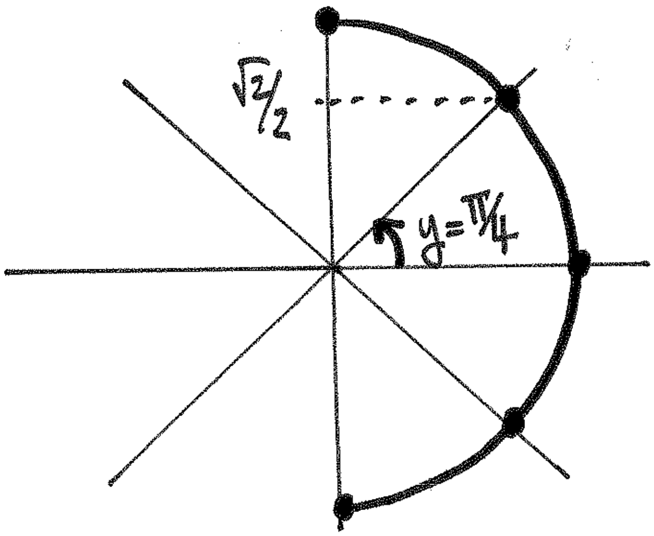
$$= \frac{1\,(\cancel{1+\sin x})}{\cos x\,(\cancel{1+\sin x})}$$

$$= \frac{1}{\cos x}$$

$$= \sec(x)$$

Therefore $\boxed{\dfrac{\cos x}{1+\sin x} + \tan x = \sec(x)}$

5(a) Put $y = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$. Then $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin(y) = \frac{\sqrt{2}}{2}$.

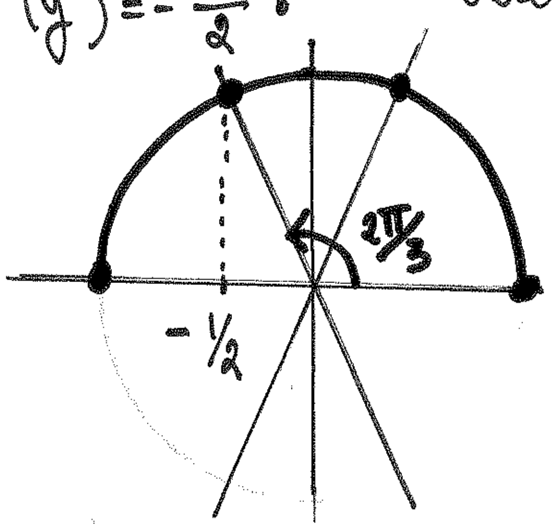


Since is the unique angle such that

$-\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ then

$$\boxed{y = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.}$$

5(b) Put $y = \cos^{-1}\left(-\frac{1}{2}\right)$. Then $y \in [0, \pi]$ and $\cos(y) = -\frac{1}{2}$.



Observe that $0 \leq \frac{2\pi}{3} \leq \pi$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$. It follows that

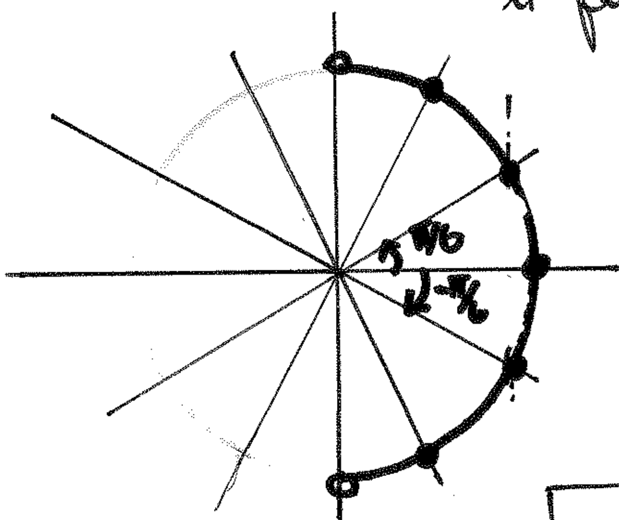
$$\boxed{y = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}}$$

5(c) Put $y = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$. Then

$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan y = -\frac{\sqrt{3}}{3}$

or $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Recalling that $\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

it follows that



$$\tan\left(-\frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}.$$

Now, since

$-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then

$$\boxed{y = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}}$$

5(d) Put $y = \tan^{-1}(1)$. Then $-\frac{\pi}{2} < y < \frac{\pi}{2}$

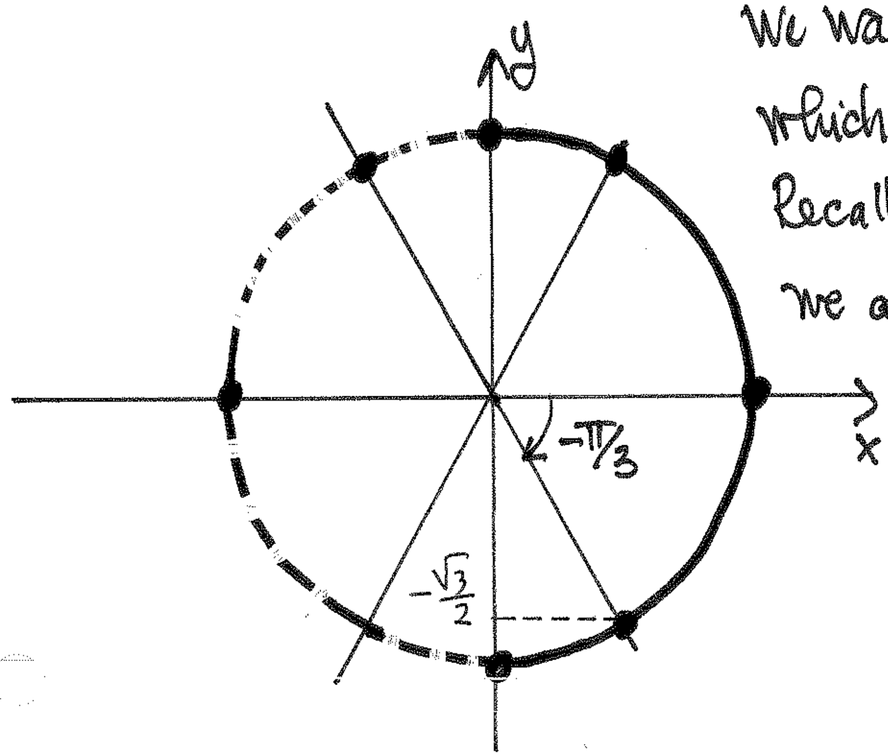
and $\tan(y) = 1$. This means that $\sin(y) = \cos(y)$

and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Therefore

$$\boxed{y = \tan^{-1}(1) = \frac{\pi}{4}}$$

5 (e) Put $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Then $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\sin y = -\frac{\sqrt{3}}{2}$ $(*)$



We want to find the angle y which satisfies $(*)$. Recalling that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, we are able to create this picture !

From the illustration above, we have

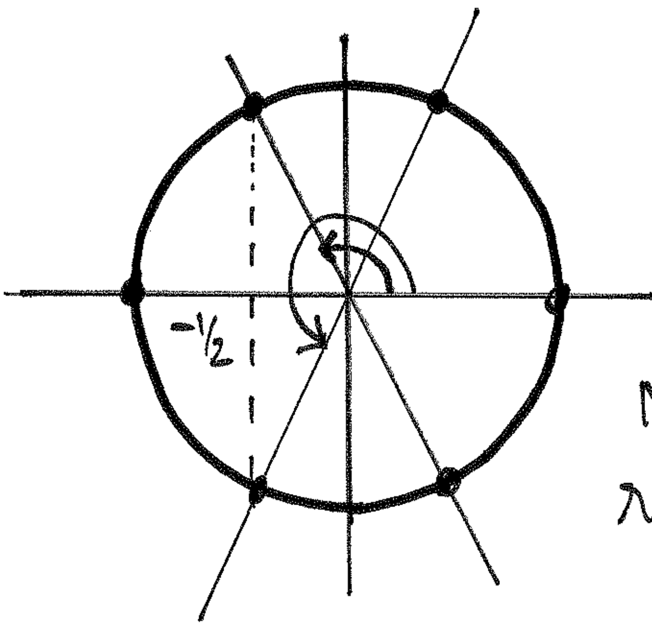
$-\frac{\pi}{2} \leq -\frac{\pi}{3} \leq \frac{\pi}{2}$ and $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

Therefore

$$\boxed{y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}}$$

6(a) Solving the equation $2\cos x = -1$.

$$2\cos x = -1 \iff \cos x = -\frac{1}{2}$$



From this picture, we observe that if $0 \leq x < 2\pi$. So,

$$x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3}.$$

Now, since there are no other restrictions for x and because the cosine function is a periodic function of period 2π then $x = \frac{2\pi}{3} + 2k\pi$ or $x = \frac{4\pi}{3} + 2k\pi$ where k is an integer. As such, the solution set is

$$\boxed{\left\{ \frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi \;\middle|\; k \in \mathbb{Z} \right\}}$$

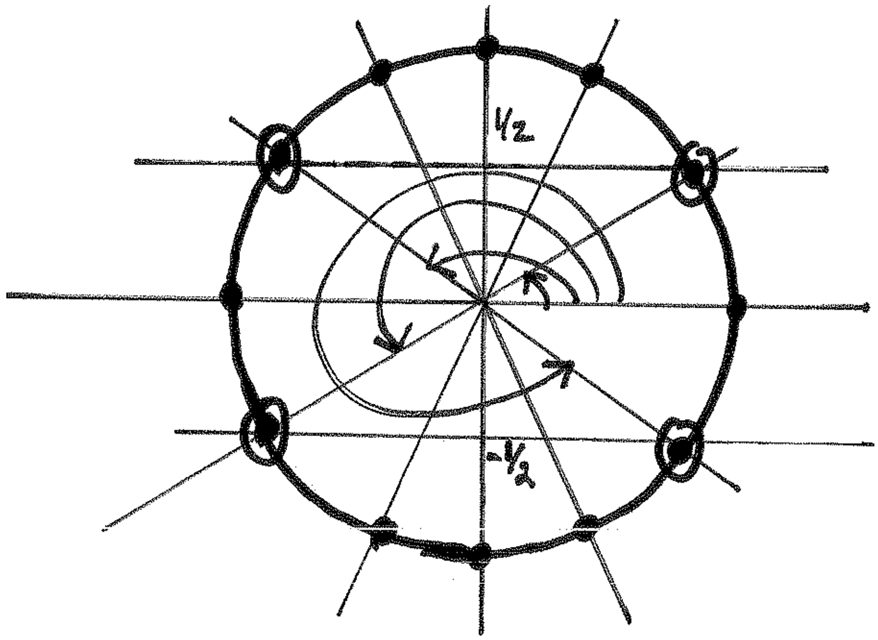
Another possible solution set is

$$\left\{ \frac{2\pi}{3} + 2k\pi, -\frac{2\pi}{3} + 2k\pi \;\middle|\; k \in \mathbb{Z} \right\}$$

6(b) Solving $4\sin^2 x = 1$.

$$4\sin^2 x = 1 \iff \sin^2 x = \frac{1}{4}$$

$$\iff \sin x = \frac{1}{2} \text{ or } \sin x = -\frac{1}{2}$$



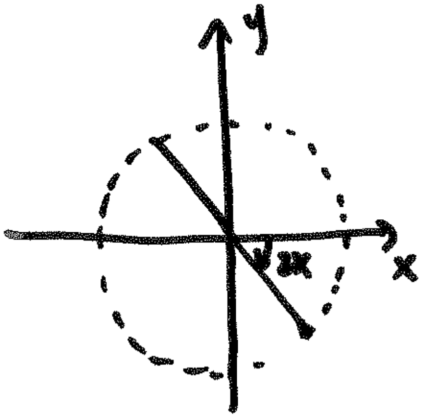
From the above, the set of solutions is given by

$$\left\{ \frac{\pi}{6} + 2k\pi,\ \frac{5\pi}{6} + 2k\pi,\ \frac{7\pi}{6} + 2k\pi,\ \frac{11\pi}{6} + 2k\pi \;\middle|\; k \in \mathbb{Z} \right\}$$

6(c) Solving $3\tan 2x = -3$, $0 \leq x < 2\pi$.

$3\tan 2x = -3 \iff \tan 2x = -1$

$\iff 2x = \tan^{-1}(-1) + k\pi \quad , k \in \mathbb{Z}$



$\iff 2x = -\frac{\pi}{4} + k\pi \quad , k \in \mathbb{Z}$

$\iff 2x = \frac{-\pi + 4k\pi}{4}$

$\iff 2x = \frac{(4k-1)\pi}{4}$

$\iff x = \frac{1}{2}\left(\frac{4k-1}{4}\right)\pi$

$\iff x = \left(\frac{4k-1}{8}\right)\pi$

Now since $0 \leq x < 2\pi$ then

$$0 \leq \left(\frac{4k-1}{8}\right)\pi < 2\pi$$

$$0 \times \frac{8}{\pi} \leq (4k-1)\frac{\pi}{8}\cdot\frac{8}{\pi} < 2\pi \cdot \frac{8}{\pi}$$

$$0 \leq 4k-1 < 16$$

$$1 \leq 4k < 17$$

$$\frac{1}{4} \leq k < \frac{17}{4}$$

Since k is an integer then

$k = 1, 2, 3$ or 4.

If $k = 1$, $x = \left(\frac{4(1)-1}{8}\right)\pi = \frac{3\pi}{8}$

If $k = 2$, then $x = \left(\frac{4(2)-1}{8}\right)\pi = \frac{7\pi}{8}$

If $k = 3$, then $x = \left(\frac{4(3)-1}{8}\right)\pi = \frac{11\pi}{8}$

If $k = 4$ then $x = \frac{4(4)-1}{8} \cdot \pi = \frac{15\pi}{8}$.

In summary, the solution set is

$$\boxed{\left\{ \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8} \right\}}$$

6(d) The equation $\cos x = \sqrt{2}$ has no solution because for any angle x

$$-1 \leq \cos x \leq 1.$$

6(e) We consider the equation

$$2\cos^2(x)\tan(x) = \tan(x), \quad 0 \leq x < 2\pi$$

Next, we observe that

$$2\cos^2(x)\tan(x) = \tan(x) \Leftrightarrow 2\cos^2 x \tan x - \tan x = 0$$

$$\Leftrightarrow (2\cos^2 x - 1)\tan x = 0$$

(we factor $\tan x$).

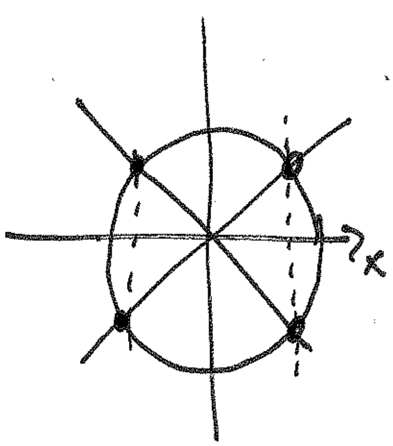
Next $(2\cos^2 x - 1)\tan x = 0 \Leftrightarrow 2\cos^2 x - 1 = 0$ or $\tan x = 0$

Now, we consider the equation $2\cos^2 x - 1 = 0$, $0 \leq x < 2\pi$

$$2\cos^2 x - 1 = 0 \Leftrightarrow \cos^2 x = \frac{1}{2}$$

$$\Leftrightarrow \cos x = \frac{1}{\sqrt{2}} \text{ or } \cos x = -\frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \cos x = \frac{\sqrt{2}}{2} \text{ or } \cos x = -\frac{\sqrt{2}}{2}$$



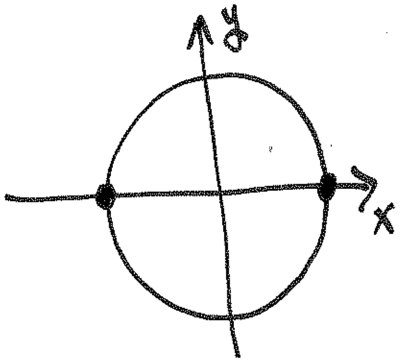
Now $2\cos^2 x - 1 = 0$ and $0 \leq x < 2\pi$ is equivalent to

$x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$, or $x = \frac{5\pi}{4}$ or

$x = \frac{7\pi}{4}$.

Next, we consider $\tan x = 0$.

$\tan x = 0 \iff \dfrac{\sin x}{\cos x} = 0 \iff \sin x = 0$



Now, since $\tan x = 0$ and $0 \le x < 2\pi$ then $x = 0$ or $x = \pi$.

In summary the set of solutions is

$$\boxed{\left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, 0, \pi \right\}}$$